Lecture 4 (preliminary version)

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Introduction

Simplicial Homology

Let $\mathbb{F}$ be a field or the ring $\mathbb{Z}$ of integers. Let $\mathcal{K}$ be a simplicial complex and let $\mathcal{K}_n$ denote the set of all n-faces of $\mathcal{K}$. Fix a total ordering $v_0 < v_1 < \ldots < v_s$ on $\mathcal{K}_0$. For each n and each $\sigma \in \mathcal{K}_n$ introduce a symbol $[\sigma] = [v_{i_0}, v_{i_1}, \ldots, v_{i_n}]$ where $\sigma_0 = \{v_{i_0}, v_{i_1}, \ldots, v_{i_n}\}$ and $i_0 < i_1 < \ldots < i_n$. Let $C_n(\mathcal{K};\mathbb{F}) = \bigoplus_{\sigma\in\mathcal{K}_n} \mathbb{F}[\sigma]$ be the $\mathbb{F}$-vector space (free abelian group if $\mathbb{F} = \mathbb{Z}$) of dimension $f_n(\mathcal{K})$ generated by the symbols $[\sigma]$. Notice that as we consider empty set as a face of $\mathcal{K}$ of dimension -1 we have $C_{-1}(\mathcal{K};\mathbb{F}) \cong \mathbb{F}$. The elements of $C_n(\mathcal{K};\mathbb{F})$ are called *n-dimensional chains* or *n-chains* of $\mathcal{K}$ and $C_n(\mathcal{K};\mathbb{F})$ itself is called the *space of n-chains*.

Boundary Maps. Now we want to define a *boundary map* $\partial_n : C_n(\mathcal{K};\mathbb{F}) \to C_{n-1}(\mathcal{K};\mathbb{F})$ that agrees with our intuition of the notion of boundary. In order to do so, it suffices to define the boundary map for generators since every element of $C_n(\mathcal{K};\mathbb{F})$ has a unique expression as linear combination of generators. We define

$$\partial_n([v_{i_0}, v_{i_1}, \ldots, v_{i_n}]) = \sum_{j=0}^{n} (-1)^j [v_{i_0}, v_{i_1}, \ldots, v_{i_{j-1}}, \hat{v}_{i_j}, v_{i_{j+1}}, \ldots, v_{i_n}],$$

where $\hat{v}_{i_j}$ denotes the omission of v_{i_j}. We also define $\partial_{-1} : C_{-1}(\mathcal{K};\mathbb{F}) \to 0$ in the obvious way.

Cycles and Boundaries An n-chain c is called an *n-cycle* if $\partial_n(c) = 0$, that is to say if $c \in \ker\, \partial_n$. The set of all n-cycles (i.e., ker ∂_n) is denoted by $Z_n(\mathcal{K};\mathbb{F})$. An n-chain c is called an *n-boundary* if $\partial_{n+1}(c') = c$ for some $(n+1)$-chain c'. The set of all n-boundaries (i.e., Im ∂_{n+1}) is denoted by $B_n(\mathcal{K};\mathbb{F})$. Clearly, $Z_n(\mathcal{K};\mathbb{F})$ and $B_n(\mathcal{K};\mathbb{F})$ inherits the algebraic structure from $C_n(\mathcal{K};\mathbb{F})$.

Lemma 1. *Let $\mathcal{K}$ be a simplicial complex. Then $\partial_n \circ \partial_{n+1} : C_{n+1}(\mathcal{K};\mathbb{F}) \to C_{n-1}(\mathcal{K};\mathbb{F})$ is the zero map. In particular $B_n(\mathcal{K};\mathbb{F}) \subseteq Z_n(\mathcal{K};\mathbb{F})$.*

Proof. It is enough to show that $\partial_n \circ \partial_{n+1}([\sigma]) = 0$ for all $\sigma \in \mathcal{K}_{n+1}$. Fix a total ordering $<$ on vertices of $\mathcal{K}$ and let $\sigma_0 = \{u_0, u_1, \ldots, u_{n+1}\}$ with $u_i < u_j$ if and only if $i < j$. Every term with non-zero coefficient in $\partial_n \circ \partial_{n+1}([\sigma])$ is of the form

$$[u_0, \ldots, u_{i-1}, \hat{u}_i, u_{i+1} \ldots, u_{j-1}, \hat{u}_j, u_{j+1}, \ldots, u_{n+1}].$$

However, this term appears exactly twice (with non-zero coefficient) in $\partial_n \circ \partial_{n+1}([\sigma])$; once with coefficient $(-1)^{i+j}$ and once with coefficient $(-1)^{i+j-1}$. □

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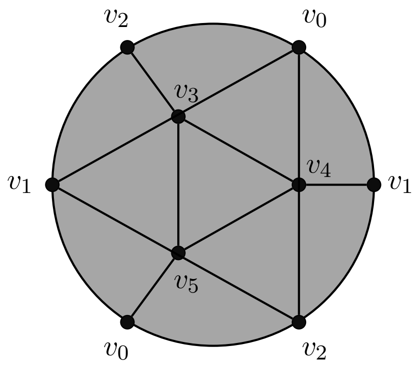


Figure 1: A Triangulation of Real Projective Plane

Reduced Simplicial Homology. The quotient $\widetilde{H}_n(\mathcal{K};\mathbb{F}) = Z_n(\mathcal{K};\mathbb{F})/B_n(\mathcal{K};\mathbb{F})$ is called the *n-th reduced simplicial homology* of $\mathcal{K}$ (with $\mathbb{F}$-coefficients). The *n-th* $\mathbb{F}$*-Betti number* (or just Betti number) $\widetilde{\beta}_n(\mathcal{K};\mathbb{F})$ of $\mathcal{K}$ is the dimension of $\dim_{\mathbb{F}} \widetilde{H}_n(\mathcal{K};\mathbb{F})$ as a vector space (or its rank as an abelian group when $\mathbb{F} = \mathbb{Z}$).

Proposition 2. *The number of connected components of a simplicial complex* $\mathcal{K}$ *is equal to* $\widetilde{\beta}_0(\mathcal{K};\mathbb{F})+1$.

Proof. □

Definition 3 (Cohen-Macaulay Complexes)**.** A simplical complex $\mathcal{K}$ is *Cohen-Macaulay* over $\mathbb{F}$ (or $\mathbb{F}$-CM) if for all faces σ of $\mathcal{K}$ one has $\widetilde{H}_i(\mathrm{lk}_{\mathcal{K}}\ \sigma;\mathbb{F}) = 0$ for all $i < \dim \mathcal{K} - \dim \sigma - 1$.

Proposition 4. *Cohen-Macaulayness depends on* $\mathbb{F}$.

Proof. □

Let $\mathcal{K}$ be a pure simplicial complex. The *dual graph* of $\mathcal{K}$ is a graph whose set of vertices are the set of facets of $\mathcal{K}$ and two vertices are connected by an edge if and only if their corresponding facets intersect in codimension one. A simplicial complex $\mathcal{K}$ is *strongly connected* if it is pure and its dual graph is connected in graph theoretic sense.

Proposition 5. *Any Cohen-Macaulay complex is strongly connected.*

Proof. □

Some Notions from Homological Algebra

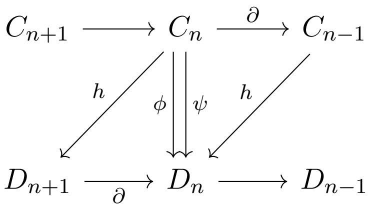
Chain Complexes. Let R be a commutative ring. A *chain complex* $\mathscr{C}$ of R-modules is a family $\{(C_n, \partial_n)\}_{n\in\mathbb{Z}}$ of R-modules C_n and R-module homomorphism $\partial_n : C_n \to C_{n-1}$ such that $\partial_n \circ \partial_{n+1} = 0$ or equivalently $Z_n := \mathrm{Im}\ \partial_{n+1} \subseteq B_n := \ker\ \partial_n$. The homomorphisms ∂_n are called *boundary maps* of *differentials* of $\mathscr{C}$. We denote the differentials always by ∂_n or simply by ∂. If we want to emphasize that ∂ is differential of $\mathscr{C}$, we write $\partial^{\mathscr{C}}$. The elements of Z_n are *cycles* and those of B_n are *boundaries.* The quotient submodule $H_n(\mathscr{C}) = Z_n/B_n$ is called the *n-th homology module* of $\mathscr{C}$. The chain complex that we constructed for a simplicial complex is called the *simplicial chain complex.*

Chain Maps. A *chain map* between chain complexes $\Phi : \mathscr{C} \to \mathscr{D}$ is a family of homomorphisms $\phi_n : C_n \to D_n$ that commutes with differentials. That is to say, every square in the diagram

$$\begin{array}{ccccccccc} \dots & \longrightarrow & C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} & \longrightarrow & \dots \\ & & \big\downarrow{\scriptstyle \phi_{n+1}} & & \big\downarrow{\scriptstyle \phi_n} & & \big\downarrow{\scriptstyle \phi_{n-1}} & & \\ \dots & \longrightarrow & D_{n+1} & \xrightarrow{\partial_{n+1}} & D_n & \xrightarrow{\partial_n} & D_{n-1} & \longrightarrow & \dots \end{array}$$

is commutative. It is easy to check that a chain map Φ takes boundaries to boundaries and cycles to cycles. Therefore, it induces a map Φ_* between homology modules.

Chain Equivalence. Two chain maps Φ and Ψ between $\mathscr{C}$ and $\mathscr{D}$ are *homotopic* if there exists a family $\mathtt{H} = \{h_n\}$ of homomorphism $h_n : C_n \to D_{n+1}$ such that $\phi_n - \psi_n = \partial_{n+1} h_n + h_{n-1}\partial_n$. These maps are illustrated in the diagram below. The maps h_n are called *chain homotopy.*



Lemma 6. *If Φ and Ψ are two homotopic chain maps from $\mathscr{C}$ to $\mathscr{D}$, then they induce the same maps on homology modules.*

Proof. For any cycle c we have $\phi(c) - \psi(c) = \partial h(c) - h\partial(c) = \partial h(c)$. Hence, $\phi(c)$ and $\psi(c)$ differ by a boundary and , consequently, they belong to the same homology class. □

Two chain complexes $\mathscr{C}$ and $\mathscr{D}$ are *chain homotopy equivalent* if there is a pair (Φ, Ψ) of chain maps $\Phi : \mathscr{C} \to \mathscr{D}$ and $\Psi : \mathscr{D} \to \mathscr{C}$ such that $\Psi\Phi$ is homotopic to the identity map of $\mathscr{C}$ and $\Phi\Psi$ is homotopic to the identity map of $\mathscr{D}$. The pair (Φ, Ψ) is called a *homotopy equivalence.*

Theorem 7. *If $\mathscr{C}$ and $\mathscr{D}$ are chain homotopy equivalent, then $H_n(\mathscr{C}) = H_n(\mathscr{D})$ for all n.*

Proof. □

Later, we give a topological intuition of chain homotopy.

Proposition 8. *If $\mathcal{K}$ and $\mathcal{L}$ are simple homotopy equivalent, then $\widetilde{H}_n(\mathcal{K}; \mathbb{F}) \cong \widetilde{H}_n(\mathcal{L}; \mathbb{F})$ for all n. In particular, If $\mathcal{K}$ is contractible, then $\mathcal{K}$ is acyclic.*

Proof. It suffices to show that if $\mathcal{L}$ is obtained by from $\mathcal{K}$ by an elementary collapse of $\tau < \sigma$, then $\widetilde{H}_n(\mathcal{K}; \mathbb{F}) \cong \widetilde{H}_n(\mathcal{L}; \mathbb{F})$. Consider the inclusion map $\iota : \mathcal{L} \to \mathcal{K}$ and the retraction $\mathtt{r} : \mathcal{K} \to \mathcal{L}$. And let $\iota_\#$ and $\mathtt{r}_\#$ be the corresponding induced maps of simplicial chain complexes. We shall show that $(\iota_\#, \mathtt{r}_\#)$ is a homotopy equivalence. Clearly $\mathtt{r}_\#\iota_\#$ is the identity on the simplicial chain complex of $\mathcal{L}$. We show that $\iota_\#\mathtt{r}_\#$ is homotopic to the identity of the chain complex of $\mathcal{K}$. □

Theorem 9. *Every shellable simplicial complex is Cohen-Macaulay.*

Proof. □

Appendix: A Topological Intuition Behind Chain Homotopy

Let I denote the closed interval $[0, 1]$. Let $\mathcal{K}$ be a piecewise linear regular cell decomposition of $\mathbb{B}^n$ and let $f(\mathcal{K})$ be a piecewise linear embedding of $\mathcal{K}$ in $\mathbb{R}^n$. Embedd $\mathbb{R}^n$ into $\mathbb{R}^{n+1}$ and by $p \to (p, 0)$. For any subcomplex $\mathcal{D}$ of $\mathcal{K}$ let $H(\mathcal{D})$ denote the multiplication $f(\mathcal{D}) \times [0, 1]$ and let $g(\mathcal{D})$ denote the intersection of $f(\mathcal{D}) \times [0, 1]$ with the hyperplane $x_{n+1} = 0$. One can think of H both as a homotopy equivalence between f and g and as a regular cell decomposition of $\mathbb{B}^{n+1}$. Let us denote the topological boundary by ∂. Then one can see that

$$\partial H(\mathcal{K}) \quad = \quad g(\mathcal{K}) \cup f(\mathcal{K}) \cup H(\partial\mathcal{K}).$$

Now in order to get the an algebraic formula, one has to look at the algebra of boundaries (i.e., the chain complexes). One can extend the incidence numbers of $\mathcal{K}$ to $H(\mathcal{K})$ as follows

$$\begin{aligned}
\varepsilon(H(\sigma), H(\tau)) &= -\varepsilon(\sigma, \tau) \\
\varepsilon(H(\sigma), f(\sigma)) &= -1 \\
\varepsilon(H(\sigma), g(\sigma)) &= 1 \\
\varepsilon(f(\sigma), f(\tau)) &= \varepsilon(\sigma, \tau) \\
\varepsilon(g(\sigma), g(\tau)) &= \varepsilon(\sigma, \tau)
\end{aligned}$$

For any subcomplex $\mathcal{D}$ set $[\mathcal{D}]$ to be the chain consisting of the summation of top dimensional faces of $\mathcal{D}$. For each σ we have

$$\begin{aligned}\partial[H(\sigma)] &= \varepsilon(H(\sigma), g(\sigma))[g(\sigma)] + \varepsilon(H(\sigma), f(\sigma))[f(\sigma)] + \sum_{\tau<\sigma} \varepsilon(H(\sigma), H(\tau))[H(\tau)] \\ &= [g(\sigma)] - [f(\sigma)] - [H(\partial\sigma)].\end{aligned}$$

Now if we take a sum over all n-dimensional faces σ of $\mathcal{K}$ we get

$$\partial[H(\mathcal{K})] = [g(\mathcal{K})] - [f(\mathcal{K})] - [H(\partial\mathcal{K})].$$